When is $\Z[\sqrt{d}]$ a unique factorization domain?

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WHEN IS $\mathbb{Z}[\sqrt{d}]$ A UNIQUE FACTORIZATION DOMAIN?

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ABSTRACT. Let d be a positive integer that is not a perfect square. In this note we give necessary and sufficient conditions for $\mathbb{Z}[\sqrt{d}]$ to be a unique factorization domain. We also apply this criterion to give an improvement of Mollin-Williams's result that provides sufficient conditions so that $\mathbb{Q}(\sqrt{d})$ has class number one.

1. INTRODUCTION

Unique factorization. A unit u in a given ring R (integral domain) is an element in R whose reciprocal $1/u$ is also in R. Two elements α, β of R are said to be associates if $\alpha = u\beta$ for some unit u, and this is an equivalence relation on R. An irreducible element in R is a nonzero, nonunit element $q \in R$ that cannot be written as the product of two non-units in R. If $\alpha, \beta \in R$ we say that α divides β, written $\alpha \mid \beta$, if there exists $\gamma \in R$ such that $\beta = \alpha\gamma$. A nonzero, nonunit element $\pi \in R$ is called a prime if $\pi \mid \alpha\beta$, where $\alpha, \beta \in R$, implies that $\pi \mid \alpha$ or $\pi \mid \beta$; equivalently, a prime is a nonzero element of R for which the principal ideal (π) is a prime ideal of R. The ring R is said to have the unique factorization property, or to be a unique factorization ring (unique factorization domain, abbreviated UFD), if every nonzero, nonunit, element in R can be expressed as a product of irreducible elements in exactly one way (where two factorizations are counted as the same if one can be obtained from the other by rearranging the order in which the irreducibles appear and multiplying them by units). In the prototype ring $\mathbb{Z}$ of ordinary integers, the only units are ± 1. The fundamental fact that any ordinary integer greater than 1 can be uniquely expressed as a product of (positive) prime numbers (that is, that $\mathbb{Z}$ enjoys the unique factorization property) is crucial for much of the number theory done with ordinary integers.

The distinction between primes and irreducibles is needed, because in a ring which is not an UFD these notions are not equivalent. Prime elements are always irreducible; the converse holds in a UFD, but not in general.

Let $d \neq 1$ be an integer that is not a perfect square, $\alpha = \frac{-1+\sqrt{d}}{2}$ if $d \equiv 1 \pmod 4$ and $\alpha = \sqrt{d}$ otherwise. We shall be concerned in collections of complex numbers that are integral linear combinations of 1 and α. These are closed under addition and multiplication, and so they form a ring, which is denoted by $\mathbb{Z}[\alpha]$. That is, $\mathbb{Z}[\alpha]$ is the set of all numbers of the form $a+b\alpha$ where

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a and b are ordinary integers numbers. A fundamental problem in number theory is to determine when $\mathbb{Z}[\alpha]$ has unique factorization property. This problem is partially solved and partially unsolved. At the 1912 International Congress of Mathematicians, Rabinowitsch [7] gave a proof of the following: Let q be a prime, let $f_q(x) = x^2 + x + q$. Then, the following assertions are equivalent:

(1) $f_q(n)$ is a prime for $n = 0, 1, ..., q - 2$.
(2) $\mathbb{Z}[\frac{-1+\sqrt{1-4q}}{2}]$ is a unique factorization domain.

Independently, Lehmer [5] proved in 1936 that $(1) \Rightarrow (2)$, while $(2) \Rightarrow (1)$ was proved again by Szekeres [13] and by Ayoub and Chowla [2]. It is now well-known (see [11]) that the ring $\mathbb{Z}[\alpha]$ for negative square-free d has unique factorization property if and only if d takes of the values:

$$-1, -2, -3, -7, -11, -19, -43, -67, -163.$$

The situation for positive d is not at all well understood. Factorization is unique in many more cases, for instance $2, 3, 5, 6, 7, 11, 13, 14, 17, 19, 21, 22,$ $23, 29, 31, 33, 37, 38, 41, 43, 46, 47, 53, 57, 59, 61, 62, 67, 69, 71, 73, 77, 83, 86, 89,$ $93, 94, 97, \ldots \ldots$ (these being all for d less than 100). It is not even known whether unique factorization occurs for infinitely many $d > 0$. Gauss conjectured that there are infinitely many real quadratic fields which have the unique factoring property. In spite of the immense amount which has been learned about quadratic fields since the time of Gauss, this conjecture seems still to be extremely difficult to prove. An interesting recent development concerning this problem is the collection of heuristics introduced by Cohen and Lenstra [4]. According these conjectures a positive fraction of all real quadratic fields will have the property of unique factorization.

For surveys of these and related results see, for instance, Alaca and Williams [1], Baker [3], Ribenboim [10], Stewart and Tall [12].

Our aim with this note is to showcase a simple proof of the following criteria for unique factorization in $\mathbb{Z}[\sqrt{d}]$. It should be pointed out that the proof of this theorem is completely elementary, not relying on ideal classes or the geometry of numbers, resting only on the commutative ring theory seen in a first undergraduate algebra course.

THEOREM 1. *Let d be a positive integer that is not a perfect square. Let $\Gamma(d)$ be the set of all primes $p \in \mathbb{N}$ satisfying $2 < p \leq \sqrt{\frac{d}{2}}$ and $\left(\frac{d}{p}\right) \neq -1$, where $\left(\frac{*}{*}\right)$ denotes the Legendre symbol. Also let $\Delta(d) = \Gamma(d) \cup \{2\}$. Then the following properties are equivalent:*

(1) *$\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*
(2) *For every $p \in \Delta(d)$, there exist integers x and y such that $p = |x^2 - dy^2|$.*

2. PROOF OF THEOREM 1

Our proof makes crucial use of the following theorem.

THEOREM 2. *Let d be a positive integer, with $d \neq 5$. Suppose that $\mathbb{Z}[\sqrt{d}]$ is not a unique factorization domain. Then, there is a prime q which is irreducible but not prime in $\mathbb{Z}[\sqrt{d}]$ such that $q \leq \sqrt{\frac{d}{2}}$.*

Proof. Put $\alpha = \sqrt{d}$. Suppose that $\mathbb{Z}[\alpha]$ is not a unique factorization domain. Then, by [8, Lemma 2.2], there is a prime number q which is not prime in $\mathbb{Z}[\alpha]$ such that

$$(1) \qquad \omega \in \mathbb{Z}[\alpha] \quad \text{and} \quad q \mid N(\omega) \qquad \text{implies that} \qquad q^2 \leq |N(\omega)|,$$

where N stands for the norm map.

Since α is a root of the polynomial $x^2 - d$ and q is not prime in $\mathbb{Z}[\alpha]$, by [8, Lemma 2.3], we get that there exists $a \in \mathbb{Z}$ such that

$$(2) \qquad 0 \leq a \leq q/2 \qquad \text{and} \qquad a^2 - d \equiv 0 \pmod{q}.$$

Let us see that

$$(3) \qquad q \leq \sqrt{\frac{d}{2}}.$$

Let $b = a - q$. Then, from (2) we obtain

$$(4) \qquad b^2 - d \equiv 0 \pmod{q},$$

and

$$(5) \qquad \frac{q}{2} \leq -b \leq q.$$

As

$$N(b - \alpha) = b^2 - d,$$

from (4) and (1) we deduce that

$$(6) \qquad q^2 \leq |N(b - \alpha)| = |b^2 - d|.$$

Combining (6) and (5), we get

$$(7) \qquad |b^2 - d| = d - b^2.$$

From (6), (7) and (5), we deduce that

$$4q^2 \leq 4d - (2b)^2 \leq 4d - q^2,$$

thus giving

$$(8) \qquad 5q^2 \leq 4d.$$

Let $c = a + q$. Then, from (2) we obtain

$$(9) \qquad c^2 - d \equiv 0 \pmod{q},$$

and

$$(10) \qquad q \leq c \leq \frac{3q}{2}.$$

As

$$N(c - \alpha) = c^2 - d,$$

from (9) and (1) we deduce that

$$(11) \qquad q^2 \le |N(c - \alpha)| = |c^2 - d|.$$

We now show that

$$(12) \qquad |c^2 - d| = d - c^2.$$

For otherwise $|c^2 - d| = c^2 - d$. From (11), (8) and (10), we get

$$4q^2 \le (2c)^2 - 4d \le 9q^2 - 5q^2 = 4q^2.$$

This forces that $4d = 5q^2$, which is impossible because $d \ne 5$. So

$$|c^2 - d| = d - c^2.$$

Combining (11), (12) and (10), we get

$$q^2 \le d - c^2 \le d - q^2,$$

giving

$$q \le \sqrt{\frac{d}{2}}.$$

To show that q is irreducible in $\mathbb{Z}[\alpha]$, first suppose that it is reducible, i.e., $q = xy$ for same non-units x, y in $\mathbb{Z}[\alpha]$, then $q^2 = N(xy) = N(x)N(y)$ with $|N(x)|, |N(y)| > 1$. Thus,

$$(13) \qquad q = |N(x)|.$$

Combining (1) and (13) we get $q^2 \le q$, which is impossible. This contradiction means that if $q = xy$ in $\mathbb{Z}[\alpha]$ then x or y is a unit in $\mathbb{Z}[\alpha]$, i.e. q is irreducible in $\mathbb{Z}[\alpha]$.

$\square$

We turn now to the proof of Theorem 1.

Proof of Theorem 1. Put $\alpha = \sqrt{d}$. Let us see that (1) $\implies$ (2). Let $p \in \Delta(d)$. It is clear that there exists $t \in \mathbb{Z}$ such that

$$(14) \qquad t^2 - d \equiv 0 \pmod{p}.$$

From (14), by [8, Lemma 2.3], we deduce that p is not prime in $\mathbb{Z}[\alpha]$. Then we can write $p = xy$ for some non-units $x, y \in \mathbb{Z}[\alpha]$. Now clearly $p^2 = N(xy) = N(x)N(y)$ with $|N(x)| > 1$ and $|N(y)| > 1$. Thus $p = |N(x)|$. Let us put $x = a + b\alpha$. Then $p = |N(x)| = |a^2 - db^2|$.

Let us now see the implication (2) $\implies$ (1). Suppose that $\mathbb{Z}[\alpha]$ is not a unique factorization domain. Since the equation $x^2 - 5y^2 = \pm 2$ has no integer solutions and $2 \in \Delta(d)$, it follows that $d \ne 5$. Then, by Theorem 2, there is a prime number p which is irreducible but not prime in $\mathbb{Z}[\alpha]$ such that

$$(15) \qquad p \le \sqrt{\frac{d}{2}}.$$

Since p is not prime in $\mathbb{Z}[\alpha]$, by [8, Lemma 2.3], we get that there exists $n \in \mathbb{Z}$ such that

(16) $$n^2 - d \equiv 0 \pmod{p}.$$

From (16) and (15) we must have $p \in \Delta(d)$ and so, according to our hypotheses there exist integers a and b such that

$$p = |a^2 - db^2| = |N(a + b\alpha)|,$$

which is impossible because p is irreducible in $\mathbb{Z}[\alpha]$. Thus, $\mathbb{Z}[\alpha]$ must be a unique factorization domain.

3. Applications

To show that our necessary and sufficient condition is not impossible to use, we present easy proofs of the following two results. We remind the reader that if R is a domain then we say that $\mathcal{R}$ is a principal ideal domain (abbreviated PID) if every ideal of $\mathcal{R}$ is principal. That is to say, given an ideal $\mathcal{I}$ of $\mathcal{R}$ there is a $\alpha \in \mathcal{R}$ such that $\mathcal{I} = \mathcal{R}\alpha$. It is now well-known (see [1, Theorem 3.3.1, page 61]) that a necessary condition for $\mathcal{R}$ to be a PID is that it be UFD.

PROPOSITION 1. *Let d be a positive integer that is not a perfect square. Consider the ring $\mathcal{R} = \mathbb{Z}[\sqrt{d}]$. If $d \equiv 1 \pmod{4}$, then $\mathcal{R}$ is not a DIP.*

Proof. Suppose that $\mathcal{R}$ is a DIP. Then $\mathcal{R}$ is a UFD, and so, by Theorem 1, we deduce that there are two odd integers x and y such that

$$2 = |x^2 - dy^2|,$$

this equation implies that

$$
\begin{aligned}
2 &\equiv x^2 - dy^2 \\
&\equiv 1 - d \equiv 0 \pmod{4},
\end{aligned}
$$

a contradiction. Thus, $\mathcal{R}$ is not a DIP. $\qquad\square$

PROPOSITION 2. *Let d be a non-square positive integer, let $t \equiv 5 \pmod{8}$ be a positive integer. Consider the ring $\mathcal{R} = \mathbb{Z}[\sqrt{d}]$. If $d \equiv 2, 3 \pmod{4}$, and if $t \mid d$, then $\mathcal{R}$ is not a DIP.*

Proof. Suppose that $\mathcal{R}$ is a DIP. Then $\mathcal{R}$ is a UFD, and so, by Theorem 1, we deduce that there are two odd integers x and y such that

$$2 = |x^2 - dy^2|,$$

since $t \mid d$ and $t \equiv 5 \pmod{8}$ this equation implies that 2 is a quadratic residue modulo t, a contradiction. Thus, $\mathcal{R}$ is not a DIP. $\qquad\square$

The following consequence of Theorem 1 is interesting in its own right.

PROPOSITION 3. *Let d be a positive integer. Suppose that d is a prime $\equiv 3 \pmod{4}$ or $d = 2q$ with q a prime $\equiv 3 \pmod{4}$. Then the following are equivalent:*

(1) $\mathbb{Z}[\sqrt{d}]$ *is a unique factorization domain.*

(2) *For every $p \in \Gamma(d)$, there exist integers x and y such that $p = |x^2 - dy^2|$*

Proof. By [14, Lemma 2.2 and Lemma 2.3] we get that the equation $2 = |x^2 - dy^2|$ is solvable in integers x, y. Therefore, proposition is obtained from Theorem 1. $\qquad\square$

COROLLARY 1. *Let d be a positive integer. Suppose that d is a prime $\equiv 3$ (mod 4) or $d = 2q$ with q a prime $\equiv 3$ (mod 4). If $\Gamma(d) = \emptyset$, then $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

THEOREM 3. *Let $d \in \{3, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398\}$. Then the ring $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

Proof. It is easily checked that $\Gamma(d) = \emptyset$ for $d = 3$, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398. Thus, by Corollary 1, we deduce that the rings $\mathbb{Z}[\sqrt{d}]$ are unique factorization domains for $d = 3$, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398. $\qquad\square$

THEOREM 4. *Let $d \in \{19, 31, 43, 59, 67, 71, 103, 107, 127, 131, 139, 151, 163, 179, 191, 199, 211, 239, 251, 263, 271, 283, 307, 311, 331, 347, 367, 379, 383, 419, 463, 467, 479, 487, 491, 503, 523, 547, 563, 571, 587, 599, 607, 619, 631, 643, 647, 683, 691, 719, 739, 743, 751, 787, 811, 823, 827, 859, 863, 883, 887, 907, 911, 919, 947, 967, 971, 983, 991\}$. Then the ring $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

Proof. Put $t = \lfloor \sqrt{\frac{d}{2}} \rfloor$ and $\Gamma = \Gamma(d)$.

If $d = 19$, then $t = 3$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(4)^2 - 19(1)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{19}\right]$ is a unique factorization domain.

If $d = 31$, then $t = 3$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(11)^2 - 31(2)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{31}\right]$ is a unique factorization domain.

If $d = 43$, then $t = 4$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(13)^2 - 43(2)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{43}\right]$ is a unique factorization domain.

If $d = 59$, then $t = 5$ and $\Gamma = \{5\}$. Furthermore, we get that
$$p = |(8)^2 - 59(1)^2|,$$
where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{59}\right]$ is a unique factorization domain.

If $d = 67$, then $t = 5$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(8)^2 - 67(1)^2|,$$

where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{67}\right]$ is a unique factorization domain.

If $d = 71$, then $t = 5$ and $\Gamma = \{5\}$. Furthermore, we get that

$$p = |(17)^2 - 71(2)^2|,$$

where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{71}\right]$ is a unique factorization domain.

If $d = 103$, then $t = 7$ and $\Gamma = \{3\}$. Furthermore, we get that

$$p = |(10)^2 - 103(1)^2|,$$

where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{103}\right]$ is a unique factorization domain.

If $d = 107$, then $t = 7$ and $\Gamma = \{7\}$. Furthermore, we get that

$$p = |(10)^2 - 107(1)^2|,$$

where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{107}\right]$ is a unique factorization domain.

If $d = 127$, then $t = 7$ and $\Gamma = \{3, 7\}$. Furthermore, we get that

$$p = |(293)^2 - 127(26)^2| \quad \text{and} \quad p' = |(45)^2 - 127(4)^2|,$$

where $p = 3$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{127}\right]$ is a unique factorization domain.

If $d = 131$, then $t = 8$ and $\Gamma = \{5\}$. Furthermore, we get that

$$p = |(23)^2 - 131(2)^2|,$$

where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{131}\right]$ is a unique factorization domain.

If $d = 139$, then $t = 8$ and $\Gamma = \{3, 5\}$. Furthermore, we get that

$$p = |(224)^2 - 139(19)^2| \quad \text{and} \quad p' = |(12)^2 - 139(1)^2|,$$

where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{139}\right]$ is a unique factorization domain.

If $d = 151$, then $t = 8$ and $\Gamma = \{3, 5, 7\}$. Furthermore, we get that

$$p = |(86)^2 - 151(7)^2|, \ p' = |(2814)^2 - 151(229)^2| \quad \text{and} \quad p'' = |(12)^2 - 151(1)^2|,$$

where $p = 3$, $p' = 5$ and $p'' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{151}\right]$ is a unique factorization domain.

If $d = 163$, then $t = 9$ and $\Gamma = \{3, 7\}$. Furthermore, we get that

$$p = |(932)^2 - 163(73)^2| \quad \text{and} \quad p' = |(51)^2 - 163(4)^2|,$$

where $p = 3$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{163}\right]$ is a unique factorization domain.

If $d = 179$, then $t = 9$ and $\Gamma = \{5, 7\}$. Furthermore, we get that

$$p = |(388)^2 - 179(29)^2| \quad \text{and} \quad p' = |(107)^2 - 179(8)^2|,$$

where $p = 5$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{179}\right]$ is a unique factorization domain.

If $d = 191$, then $t = 9$ and $\Gamma = \{5, 7\}$. Furthermore, we get that
$$p = |(14)^2 - 191(1)^2| \quad \text{and} \quad p' = |(152)^2 - 191(11)^2|,$$
where $p = 5$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{191}\right]$ is a unique factorization domain.

If $d = 199$, then $t = 9$ and $\Gamma = \{3, 5\}$. Furthermore, we get that
$$p = |(14)^2 - 199(1)^2| \quad \text{and} \quad p' = |(2511)^2 - 199(178)^2|,$$
where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{199}\right]$ is a unique factorization domain.

If $d = 211$, then $t = 10$ and $\Gamma = \{3, 5, 7\}$. Furthermore, we get that
$$p = |(29)^2 - 211(2)^2|, \ p' = |(276)^2 - 211(19)^2| \quad \text{and} \quad p'' = |(125082)^2 - 211(8611)^2|,$$
where $p = 3$, $p' = 5$ and $p'' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{211}\right]$ is a unique factorization domain.

If $d = 239$, then $t = 10$ and $\Gamma = \{5, 7\}$. Furthermore, we get that
$$p = |(31)^2 - 239(2)^2| \quad \text{and} \quad p' = |(572)^2 - 239(37)^2|,$$
where $p = 5$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{239}\right]$ is a unique factorization domain.

If $d = 251$, then $t = 11$ and $\Gamma = \{5, 11\}$. Furthermore, we get that
$$p = |(16)^2 - 251(1)^2| \quad \text{and} \quad p' = |(95)^2 - 251(6)^2|,$$
where $p = 5$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{251}\right]$ is a unique factorization domain.

If $d = 263$, then $t = 11$ and $\Gamma = \{7\}$. Furthermore, we get that
$$p = |(16)^2 - 263(1)^2|,$$
where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{263}\right]$ is a unique factorization domain.

If $d = 271$, then $t = 11$ and $\Gamma = \{3, 5\}$. Furthermore, we get that
$$p = |(214)^2 - 271(13)^2| \quad \text{and} \quad p' = |(33)^2 - 271(2)^2|,$$
where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{271}\right]$ is a unique factorization domain.

If $d = 283$, then $t = 11$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(1043)^2 - 283(62)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{283}\right]$ is a unique factorization domain.

If $d = 307$, then $t = 12$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(35)^2 - 307(2)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{307}\right]$ is a unique factorization domain.

If $d = 311$, then $t = 12$ and $\Gamma = \{5, 11\}$. Furthermore, we get that

$$p = |(194)^2 - 311(11)^2| \quad \text{and} \quad p' = |(1305)^2 - 311(74)^2|,$$

where $p = 5$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{311}\right]$ is a unique factorization domain.

If $d = 331$, then $t = 12$ and $\Gamma = \{3, 5, 7, 11\}$. Furthermore, we get that

$$p = |(870136)^2 - 331(47827)^2|, \ p' = |(18)^2 - 331(1)^2|,$$

$$p'' = |(18)^2 - 331(1)^2| \quad \text{and} \quad p''' = |(65660)^2 - 331(3609)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{331}\right]$ is a unique factorization domain.

If $d = 347$, then $t = 13$ and $\Gamma = \{7, 13\}$. Furthermore, we get that

$$p = |(149)^2 - 347(8)^2| \quad \text{and} \quad p' = |(56)^2 - 347(3)^2|,$$

where $p = 7$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{347}\right]$ is a unique factorization domain.

If $d = 367$, then $t = 13$ and $\Gamma = \{3, 11, 13\}$. Furthermore, we get that

$$p = |(10843)^2 - 367(566)^2|, \ p' = |(3046)^2 - 367(159)^2| \quad \text{and}$$

$$p'' = |(115)^2 - 367(6)^2|,$$

where $p = 3$, $p' = 11$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{367}\right]$ is a unique factorization domain.

If $d = 379$, then $t = 13$ and $\Gamma = \{3, 5, 7, 11\}$. Furthermore, we get that

$$p = |(33076)^2 - 379(1699)^2|, \ p' = |(39)^2 - 379(2)^2|,$$

$$p'' = |(435147)^2 - 379(22352)^2| \quad \text{and} \quad p''' = |(292)^2 - 379(15)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{379}\right]$ is a unique factorization domain.

If $d = 383$, then $t = 13$ and $\Gamma = \{11\}$. Furthermore, we get that

$$p = |(39)^2 - 383(2)^2|,$$

where $p = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{383}\right]$ is a unique factorization domain.

If $d = 419$, then $t = 14$ and $\Gamma = \{5, 11, 13\}$. Furthermore, we get that

$$p = |(41)^2 - 419(2)^2|, \ p' = |(4360)^2 - 419(213)^2| \quad \text{and}$$

$$p'' = |(348)^2 - 419(17)^2|,$$

where $p = 5$, $p' = 11$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{419}\right]$ is a unique factorization domain.

If $d = 463$, then $t = 15$ and $\Gamma = \{3, 7, 11\}$. Furthermore, we get that

$$p = |(43)^2 - 463(2)^2|, \ p' = |(624)^2 - 463(29)^2| \quad \text{and}$$

$$p'' = |(4070281)^2 - 463(189162)^2|,$$

where $p = 3$, $p' = 7$ and $p'' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{463}\right]$ is a unique factorization domain.

If $d = 467$, then $t = 15$ and $\Gamma = \{11, 13\}$. Furthermore, we get that
$$p = |(108)^2 - 467(5)^2| \quad \text{and} \quad p' = |(389)^2 - 467(18)^2|,$$
where $p = 11$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{467}\right]$ is a unique factorization domain.

If $d = 479$, then $t = 15$ and $\Gamma = \{5\}$. Furthermore, we get that
$$p = |(22)^2 - 479(1)^2|,$$
where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{479}\right]$ is a unique factorization domain.

If $d = 487$, then $t = 15$ and $\Gamma = \{3, 7, 11\}$. Furthermore, we get that
$$p = |(22)^2 - 487(1)^2|, \ p' = |(76179)^2 - 487(3452)^2| \quad \text{and}$$
$$p'' = |(9401)^2 - 487(426)^2|,$$
where $p = 3$, $p' = 7$ and $p'' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{487}\right]$ is a unique factorization domain.

If $d = 491$, then $t = 15$ and $\Gamma = \{5, 7, 13\}$. Furthermore, we get that
$$p = |(1817)^2 - 491(82)^2|, \ p' = |(22)^2 - 491(1)^2| \quad \text{and}$$
$$p'' = |(133)^2 - 491(6)^2|,$$
where $p = 5$, $p' = 7$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{491}\right]$ is a unique factorization domain.

If $d = 503$, then $t = 15$ and $\Gamma = \{13\}$. Furthermore, we get that
$$p = |(45)^2 - 503(2)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{503}\right]$ is a unique factorization domain.

If $d = 523$, then $t = 16$ and $\Gamma = \{3, 13\}$. Furthermore, we get that
$$p' = |(18707)^2 - 523(818)^2| \quad \text{and} \quad p' = |(5420)^2 - 503(237)^2|,$$
where $p = 3$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{523}\right]$ is a unique factorization domain.

If $d = 547$, then $t = 16$ and $\Gamma = \{3, 7, 13\}$. Furthermore, we get that
$$p = |(12115)^2 - 547(518)^2|, \ p' = |(378699)^2 - 547(16192)^2| \quad \text{and}$$
$$p'' = |(421)^2 - 547(18)^2|,$$
where $p = 3$, $p' = 7$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{547}\right]$ is a unique factorization domain.

If $d = 563$, then $t = 16$ and $\Gamma = \{13\}$. Furthermore, we get that
$$p = |(24)^2 - 563(1)^2|,$$
where $p = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{563}\right]$ is a unique factorization domain.

If $d = 571$, then $t = 16$ and $\Gamma = \{3, 5, 7, 13\}$. Furthermore, we get that
$$p = |(4349)^2 - 571(182)^2|, \ p' = |(24)^2 - 571(1)^2|,$$

$p'' = |(692877)^2 - 571(28996)^2|$ and $p''' = |(406819973)^2 - 571(17024886)^2|$,
where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{571}\right]$ is a unique factorization domain.

If $d = 587$, then $t = 17$ and $\Gamma = \{11, 17\}$. Furthermore, we get that
$$p' = |(24)^2 - 587(1)^2| \quad \text{and} \quad p' = |(97)^2 - 587(4)^2|,$$
where $p = 11$ and $p' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{587}\right]$ is a unique factorization domain.

If $d = 599$, then $t = 17$ and $\Gamma = \{5, 7, 11, 13, 17\}$. Furthermore, we get that
$$p = |(49)^2 - 599(2)^2|,\ p' = |(37397)^2 - 599(1528)^2|,\ p'' = |(8150)^2 - 599(333)^2|,$$
$$p''' = |(1305542)^2 - 599(53343)^2| \quad \text{and} \quad q = |(1444)^2 - 599(59)^2|,$$
where $p = 5$, $p' = 7$, $p'' = 11$ $p''' = 13$ and $q = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{599}\right]$ is a unique factorization domain.

If $d = 607$, then $t = 17$ and $\Gamma = \{3, 13\}$. Furthermore, we get that
$$p' = |(2242)^2 - 607(91)^2| \quad \text{and} \quad p' = |(74)^2 - 607(3)^2|,$$
where $p = 3$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{607}\right]$ is a unique factorization domain.

If $d = 619$, then $t = 17$ and $\Gamma = \{3, 5, 11\}$. Furthermore, we get that
$$p = |(43254596)^2 - 619(1738549)^2|,\ p' = |(543348)^2 - 619(21839)^2| \quad \text{and}$$
$$p'' = |(17167)^2 - 619(690)^2|,$$
where $p = 3$, $p' = 5$ and $p'' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{619}\right]$ is a unique factorization domain.

If $d = 631$, then $t = 17$ and $\Gamma = \{3, 5, 7, 11, 17\}$. Furthermore, we get that
$$p = |(13244646374)^2 - 631(527261047)^2|,\ p' = |(3567)^2 - 631(142)^2|,$$
$$p'' = |(322512)^2 - 631(12839)^2|,\ p''' = |(35042)^2 - 631(1395)^2| \quad \text{and}$$
$$q = |(201)^2 - 631(8)^2|,$$
where $p = 3$, $p' = 5$, $p'' = 7$ $p''' = 11$ and $q = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{631}\right]$ is a unique factorization domain.

If $d = 643$, then $t = 17$ and $\Gamma = \{3, 11\}$. Furthermore, we get that
$$p' = |(355)^2 - 643(14)^2| \quad \text{and} \quad p' = |(76)^2 - 643(3)^2|,$$
where $p = 3$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{643}\right]$ is a unique factorization domain.

If $d = 647$, then $t = 17$ and $\Gamma = \{11, 13, 17\}$. Furthermore, we get that
$$p = |(2391)^2 - 647(94)^2|,\ p' = |(51)^2 - 647(2)^2| \quad \text{and}$$
$$p'' = |(407)^2 - 647(16)^2|,$$
where $p = 11$, $p' = 13$ and $p'' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{647}\right]$ is a unique factorization domain.

If $d = 683$, then $t = 18$ and $\Gamma = \{7, 11\}$. Furthermore, we get that
$$p = |(26)^2 - 683(1)^2| \quad \text{and} \quad p' = |(392)^2 - 683(15)^2|,$$

where $p = 7$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{683}\right]$ is a unique factorization domain.

If $d = 691$, then $t = 18$ and $\Gamma = \{3, 5, 11\}$. Furthermore, we get that
$$p = |(184)^2 - 691(7)^2|, \; p' = |(3207)^2 - 691(122)^2| \quad \text{and}$$
$$p'' = |(388834592)^2 - 691(14791965)^2|,$$
where $p = 3$, $p' = 5$ and $p'' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{691}\right]$ is a unique factorization domain.

If $d = 719$, then $t = 18$ and $\Gamma = \{5, 11, 13\}$. Furthermore, we get that
$$p = |(59018)^2 - 719(2201)^2|, \; p' = |(3030)^2 - 719(113)^2| \quad \text{and}$$
$$p'' = |(13997)^2 - 719(522)^2|,$$
where $p = 5$, $p' = 11$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{719}\right]$ is a unique factorization domain.

If $d = 739$, then $t = 19$ and $\Gamma = \{3, 5, 7, 17, 19\}$. Furthermore, we get that
$$p = |(6841564)^2 - 739(251671)^2|, \; p' = |(28889646951)^2 - 739(1062722842)^2|,$$
$$p'' = |(620125683)^2 - 739(22811692)^2|, p''' = |(20769)^2 - 739(764)^2| \quad \text{and}$$
$$q = |(597000)^2 - 739(21961)^2|,$$
where $p = 3$, $p' = 5$, $p'' = 7$ $p''' = 17$ and $q = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{739}\right]$ is a unique factorization domain.

If $d = 743$, then $t = 19$ and $\Gamma = \{7\}$. Furthermore, we get that
$$p = |(109)^2 - 743(4)^2|,$$
where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{743}\right]$ is a unique factorization domain.

If $d = 751$, then $t = 19$ and $\Gamma = \{3, 5, 7, 11, 13\}$. Furthermore, we get that
$$p = |(29566408741)^2 - 751(1078893578)^2|, \; p' = |(57388086)^2 - 751(2094121)^2|,$$
$$p'' = |(1884775533)^2 - 751(68776436)^2|, \; p''' = |(546036173942)^2 - 751(19925142975)^2|$$
$$\text{and} \quad q = |(1943683)^2 - 751(70926)^2|,$$
where $p = 3$, $p' = 5$, $p'' = 7$ $p''' = 11$ and $q = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{751}\right]$ is a unique factorization domain.

If $d = 787$, then $t = 19$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(28)^2 - 787(1)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{787}\right]$ is a unique factorization domain.

If $d = 811$, then $t = 20$ and $\Gamma = \{3, 5\}$. Furthermore, we get that
$$p = |(2692259)^2 - 811(94538)^2| \quad \text{and} \quad p' = |(57)^2 - 811(2)^2|,$$
where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{811}\right]$ is a unique factorization domain.

If $d = 823$, then $t = 20$ and $\Gamma = \{3, 7, 11, 13, 19\}$. Furthermore, we get that
$$p = |(470282)^2 - 823(16393)^2|, \; p' = |(459)^2 - 823(16)^2|,$$

$$p'' = |(86)^2 - 823(3)^2|, \ p''' = |(8886445)^2 - 823(309762)^2| \quad \text{and}$$
$$q = |(36016062)^2 - 823(1255441)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$ $p''' = 13$ and $q = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{823}\right]$ is a unique factorization domain.

If $d = 827$, then $t = 20$ and $\Gamma = \{7\}$. Furthermore, we get that

$$p = |(115)^2 - 827(4)^2|,$$

where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{827}\right]$ is a unique factorization domain.

If $d = 859$, then $t = 20$ and $\Gamma = \{3, 5, 11, 13, 17, 19\}$. Furthermore, we get that

$$p = |(332771)^2 - 859(11354)^2|, \ p' = |(9414978999)^2 - 859(321234938)^2|,$$
$$p'' = |(442140817)^2 - 859(15085650)^2|, \ p''' = |(88)^2 - 859(3)^2|,$$
$$q = |(65010921)^2 - 859(2218144)^2| \quad \text{and} \quad q' = |(6741)^2 - 859(230)^2|, \ ,$$

where $p = 3$, $p' = 5$, $p'' = 11$, $p''' = 13$, $q = 17$ and $q' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{859}\right]$ is a unique factorization domain.

If $d = 863$, then $t = 20$ and $\Gamma = \{7, 11, 17, \}$. Furthermore, we get that

$$p = |(235)^2 - 863(8)^2|, \ p' = |(25646)^2 - 863(873)^2| \quad \text{and}$$
$$p'' = |(7873)^2 - 863(268)^2|,$$

where $p = 7$, $p' = 11$ and $p'' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{863}\right]$ is a unique factorization domain.

If $d = 883$, then $t = 21$ and $\Gamma = \{3, 7, 11, 13, 17, 19\}$. Furthermore, we get that

$$p = |(208)^2 - 883(7)^2|, \ p' = |(160563114)^2 - 883(5403379)^2|,$$
$$p'' = |(16492)^2 - 883(555)^2|, p''' = |(13628440204)^2 - 883(458633529)^2|,$$
$$q = |(30)^2 - 883(1)^2| \quad \text{and} \quad q' = |(2851452)^2 - 883(95959)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$, $p''' = 13$, $q = 17$ and $q' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{883}\right]$ is a unique factorization domain.

If $d = 887$, then $t = 21$ and $\Gamma = \{13\}$. Furthermore, we get that

$$p = |(30)^2 - 887(1)^2|,$$

where $p = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{887}\right]$ is a unique factorization domain.

If $d = 907$, then $t = 21$ and $\Gamma = \{3, 7, 11, 13, \}$. Furthermore, we get that

$$p = |(4397)^2 - 907(146)^2|, \ p' = |(30)^2 - 907(1)^2|, \ p'' = |(706351)^2 - 907(23454)^2| \quad \text{and}$$
$$p''' = |(266874460)^2 - 907(8861421)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$ and $p''' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{907}\right]$ is a unique factorization domain.

If $d = 911$, then $t = 21$ and $\Gamma = \{5, 7, 11, 13\}$. Furthermore, we get that

$$p = |(17174)^2 - 911(569)^2|, \ p' = |(332)^2 - 911(11)^2|,$$

$$p'' = |(30)^2 - 911(1)^2| \quad \text{and} \quad p''' = |(830298)^2 - 911(27509)^2|,$$

where $p = 5$, $p' = 7$, $p'' = 11$ and $p''' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{911}\right]$ is a unique factorization domain.

If $d = 919$, then $t = 21$ and $\Gamma = \{3, 5, 7, 13, 17, 19\}$. Furthermore, we get that

$$p = |(260891)^2 - 919(8606)^2|, \ p' = |(1839697758129)^2 - 919(60686029438)^2|,$$

$$p'' = |(3933366603)^2 - 919(129749792)^2|, \ p''' = |(84358253)^2 - 919(2782722)^2|,$$

$$q = |(576)^2 - 919(19)^2| \quad \text{and} \quad q' = |(30)^2 - 919(1)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ $p''' = 13$ $q = 17$ and $q' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{919}\right]$ is a unique factorization domain.

If $d = 947$, then $t = 21$ and $\Gamma = \{7, 11, 19\}$. Furthermore, we get that

$$p = |(13171)^2 - 947(428)^2|, \ p' = |(2308)^2 - 947(75)^2| \quad \text{and}$$

$$p'' = |(677)^2 - 947(22)^2|,$$

where $p = 7$, $p' = 11$ and $p'' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{947}\right]$ is a unique factorization domain.

If $d = 967$, then $t = 21$ and $\Gamma = \{3, 7, 17, 19\}$. Furthermore, we get that

$$p = |(104173658)^2 - 967(3349999)^2|, \ p' = |(1172436)^2 - 967(37703)^2|,$$

$$p'' = |(26712)^2 - 967(859)^2| \quad \text{and} \quad p''' = |(31369341)^2 - 967(1008770)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 17$ and $p''' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{947}\right]$ is a unique factorization domain.

If $d = 971$, then $t = 22$ and $\Gamma = \{5, 11, 13, 17\}$. Furthermore, we get that

$$p = |(284156)^2 - 971(9119)^2|, \ p' = |(7167)^2 - 971(230)^2|,$$

$$p'' = |(187)^2 - 971(6)^2| \quad \text{and} \quad p''' = |(38546)^2 - 971(1237)^2|,$$

where $p = 5$, $p' = 11$, $p'' = 13$ and $p''' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{971}\right]$ is a unique factorization domain.

If $d = 983$, then $t = 22$ and $\Gamma = \{11\}$. Furthermore, we get that

$$p = |(94)^2 - 983(3)^2|,$$

where $p = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{983}\right]$ is a unique factorization domain.

If $d = 991$, then $t = 22$ and $\Gamma = \{3, 5, 7, 11, 13\}$. Furthermore, we get that

$$p = |(1174169376379)^2 - 991(37298719478)^2|, \ p' = |(63)^2 - 991(2)^2|,$$

$$p'' = |(99653067)^2 - 991(3165584)^2|, \ p''' = |(203463013810)^2 - 991(6463215639)^2|$$

$$\text{and} \quad q = |(875240822)^2 - 991(27802941)^2||,$$

where $p = 3$, $p' = 5$, $p'' = 7$ $p''' = 11$ and $q = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{991}\right]$ is a unique factorization domain.

$\square$

THEOREM 5. *Let* $d \in \{2, 22, 46, 59, 86, 94, 118,\ 134, 158, 166, 206, 214, 262,$
$278, 302, 334, 358, 382, 422, 446, 454, 478, 502, 526, 542, 566, 614, 622, 662, 694, 718,$
$734, 758, 766, 838, 862, 878, 886, 926, 958, 974, 998\}$. *Then the ring* $\mathbb{Z}[\sqrt{d}]$ *is a unique factorization domain.*

Proof. Put $t = \lfloor \sqrt{\frac{d}{2}} \rfloor$ and $\Gamma = \Gamma(d)$.

If $d = 2$, then $t = 1$ and $\Delta(d) = \{2\}$. Furthermore, we get that
$$p = |(2)^2 - 2(1)^2|,$$
where $p = 2$. By Theorem 1, we get that $\mathbb{Z}\left[\sqrt{2}\right]$ is a unique factorization domain.

If $d = 22$ then $t = 3$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(5)^2 - 22(1)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{22}\right]$ is a unique factorization domain.

If $d = 46$, then $t = 4$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(7)^2 - 46(1)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{46}\right]$ is a unique factorization domain.

If $d = 86$, then $t = 6$ and $\Gamma = \{5\}$. Furthermore, we get that
$$p = |(9)^2 - 86(1)^2|,$$
where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{86}\right]$ is a unique factorization domain.

If $d = 94$, then $t = 6$ and $\Gamma = \{3, 5\}$. Furthermore, we get that
$$p = |(223)^2 - 94(23)^2| \quad \text{and} \quad p' = |(29)^2 - 94(3)^2|,$$
where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{94}\right]$ is a unique factorization domain.

If $d = 118$, then $t = 7$ and $\Gamma = \{3\}$. Furthermore, we get that
$$p = |(11)^2 - 118(1)^2|,$$
where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{118}\right]$ is a unique factorization domain.

If $d = 134$, then $t = 8$ and $\Gamma = \{5, 7\}$. Furthermore, we get that
$$p = |(81)^2 - 134(7)^2| \quad \text{and} \quad p' = |(23)^2 - 134(2)^2|,$$
where $p = 5$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{134}\right]$ is a unique factorization domain.

If $d = 158$, then $t = 8$ and $\Gamma = \{7\}$. Furthermore, we get that
$$p = |(25)^2 - 158(2)^2|,$$
where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{158}\right]$ is a unique factorization domain.

If $d = 166$, then $t = 9$ and $\Gamma = \{3, 5\}$. Furthermore, we get that

$$p = |(13)^2 - 166(1)^2| \quad \text{and} \quad p' = |(889)^2 - 166(69)^2|,$$

where $p = 3$ and $p' = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{166}\right]$ is a unique factorization domain.

If $d = 206$, then $t = 10$ and $\Gamma = \{5\}$. Furthermore, we get that

$$p = |(43)^2 - 206(3)^2|,$$

where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{206}\right]$ is a unique factorization domain.

If $d = 214$, then $t = 10$ and $\Gamma = \{3, 5, 7\}$. Furthermore, we get that

$$p = |(2443)^2 - 214(167)^2|, \ p' = |(47441)^2 - 214(3243)^2| \quad \text{and}$$

$$p'' = |(117)^2 - 214(8)^2|,$$

where $p = 3$, $p' = 5$ and $p'' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{214}\right]$ is a unique factorization domain.

If $d = 262$, then $t = 11$ and $\Gamma = \{3, 11\}$. Furthermore, we get that

$$p = |(955)^2 262 - 262(59)^2| \quad \text{and} \quad p' = |(81)^2 - 262(5)^2|,$$

where $p = 3$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{262}\right]$ is a unique factorization domain.

If $d = 278$, then $t = 11$ and $\Gamma = \{11\}$. Furthermore, we get that

$$p = |(17)^2 - 278(1)^2|,$$

where $p = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{278}\right]$ is a unique factorization domain.

If $d = 302$, then $t = 12$ and $\Gamma = \{7, 11\}$. Furthermore, we get that

$$p = |(139)^2 - 302(8)^2| \quad \text{and} \quad p' = |(643)^2 - 302(37)^2|,$$

where $p = 7$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{302}\right]$ is a unique factorization domain.

If $d = 334$, then $t = 12$ and $\Gamma = \{3, 5, 11\}$. Furthermore, we get that

$$p = |(22607)^2 - 334(1237)^2|, \ p' = |(1100981)^2 - 334(60243)^2| \quad \text{and}$$

$$p'' = |(3381)^2 - 334(185)^2|,$$

where $p = 3$, $p' = 5$ and $p'' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{334}\right]$ is a unique factorization domain.

If $d = 358$, then $t = 13$ and $\Gamma = \{7\}$. Furthermore, we get that

$$p = |(5979)^2 - 358(316)^2|,$$

where $p = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{358}\right]$ is a unique factorization domain.

If $d = 382$, then $t = 13$ and $\Gamma = \{3, 7\}$. Furthermore, we get that

$$p = |(215)^2 - 382(11)^2| \quad \text{and} \quad p' = |(39)^2 - 382(2)^2|,$$

where $p = 3$ and $p' = 7$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{382}\right]$ is a unique factorization domain.

If $d = 422$, then $t = 14$ and $\Gamma = \{7, 11\}$. Furthermore, we get that

$$p = |(41)^2 - 422(2)^2| \quad \text{and} \quad p' = |(719)^2 - 422(35)^2|,$$

where $p = 7$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{422}\right]$ is a unique factorization domain.

If $d = 446$, then $t =$ and $\Gamma = \{5, 13\}$. Furthermore, we get that

$$p = |(21)^2 - 446(1)^2| \quad \text{and} \quad p' = |(359)^2 - 446(17)^2|,$$

where $p = 5$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{446}\right]$ is a unique factorization domain.

If $d = 454$, then $t = 15$ and $\Gamma = \{3, 5, 11, 13\}$. Furthermore, we get that

$$p = |(277)^2 - 454(13)^2|, \; p' = |(5083639)^2 - 454(238587)^2|,$$

$$p'' = |(7905)^2 - 454(371)^2| \quad \text{and} \quad p''' = |(21)^2 - 454(1)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 11$ and $p''' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{454}\right]$ is a unique factorization domain.

If $d = 478$, then $t = 15$ and $\Gamma = \{3, 7, 11, 13\}$. Furthermore, we get that

$$p = |(2750545)^2 - 478(125807)^2|, \; p' = |(4635)^2 - 478(212)^2|,$$

$$p'' = |(86469)^2 - 478(3955)^2| \quad \text{and} \quad p''' = |(153)^2 - 478(7)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$ and $p''' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{478}\right]$ is a unique factorization domain.

If $d = 502$, then $t = 15$ and $\Gamma = \{3\}$. Furthermore, we get that

$$p = |(829)^2 - 502(37)^2|,$$

where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{502}\right]$ is a unique factorization domain.

If $d = 526$, then $t = 16$ and $\Gamma = \{3, 5, 7, 11\}$. Furthermore, we get that

$$p = |(23)^2 - 526(1)^2|, \; p' = |(327989)^2 - 526(14301)^2|,$$

$$p'' = |(93278457)^2 - 526(4067134)^2| \quad \text{and} \quad p'' = |(5619)^2 - 526(245)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{526}\right]$ is a unique factorization domain.

If $d = 542$, then $t = 16$ and $\Gamma = \{11, 13\}$. Furthermore, we get that

$$p = |(163)^2 - 542(7)^2| \quad \text{and} \quad p' = |(23)^2 - 542(1)^2|,$$

where $p = 11$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{542}\right]$ is a unique factorization domain.

If $d = 566$, then $t = 16$ and $\Gamma = \{5, 11\}$. Furthermore, we get that

$$p = |(1023)^2 - 566(43)^2| \quad \text{and} \quad p' = |(119)^2 - 566(5)^2|,$$

where $p = 5$ and $p' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{566}\right]$ is a unique factorization domain.

If $d = 614$, then $t = 17$ and $\Gamma = \{5, 11, 13, 17\}$. Furthermore, we get that

$$p = |(223)^2 - 614(9)^2|, \ p' = |(25)^2 - 614(1)^2|,$$

$$p'' = |(30949)^2 - 614(1249)^2| \quad \text{and} \quad p''' = |(2131)^2 - 614(86)^2|,$$

where $p = 5$, $p' = 11$, $p'' = 13$ and $p''' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{614}\right]$ is a unique factorization domain.

If $d = 622$, then $t = 17$ and $\Gamma = \{3\}$. Furthermore, we get that

$$p = |(25)^2 - 622(1)^2|,$$

where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{622}\right]$ is a unique factorization domain.

If $d = 662$, then $t = 18$ and $\Gamma = \{7, 13, 17\}$. Furthermore, we get that

$$p = |(5609)^2 - 662(218)^2|, \ p' = |(283)^2 - 662(11)^2| \quad \text{and}$$

$$p'' = |(103)^2 - 662(4)^2|,$$

where $p = 7$, $p' = 13$ and $p'' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{662}\right]$ is a unique factorization domain.

If $d = 694$, then $t = 18$ and $\Gamma = \{3, 5, 7, 11\}$. Furthermore, we get that

$$p = |(1605317)^2 - 694(60937)^2|, \ p' = |(79)^2 - 694(3)^2|,$$

$$p'' = |(843)^2 - 694(32)^2| \quad \text{and} \quad p''' = |(56554119)^2 - 694(2146765)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 11$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{694}\right]$ is a unique factorization domain.

If $d = 718$, then $t = 18$ and $\Gamma = \{3, 7, 11, 13, 17\}$. Furthermore, we get that

$$p = |(33460399)^2 - 718(1248731)^2|, \ p' = |(1179)^2 - 718(44)^2|,$$

$$p'' = |(27)^2 - 718(1)^2|, \ p''' = |(2030913)^2 - 718(75793)^2| \quad \text{and}$$

$$q = |(216615)^2 - 718(8084)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$, $p''' = 13$ and $q = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{718}\right]$ is a unique factorization domain.

If $d = 734$, then $t = 19$ and $\Gamma = \{5\}$. Furthermore, we get that

$$p = |(27)^2 - 734(1)^2|,$$

where $p = 5$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{734}\right]$ is a unique factorization domain.

If $d = 758$, then $t = 19$ and $\Gamma = \{7, 13, 19\}$. Furthermore, we get that

$$p = |(55)^2 - 758(2)^2|, \ p' = |(4763)^2 - 758(173)^2| \quad \text{and} \quad p'' = |(413)^2 - 758(15)^2|,$$

where $p = 7$, $p' = 13$ and $p'' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{758}\right]$ is a unique factorization domain.

If $d = 766$, then $t = 19$ and $\Gamma = \{3, 5, 13, 17, 19\}$. Furthermore, we get that

$$p = |(31247)^2 - 766(1129)^2|, \ p' = |(83)^2 - 766(3)^2|,$$

$$p'' = |(3681)^2 - 766(133)^2|, \ p''' = |(37475421)^2 - 766(1354042)^2| \quad \text{and}$$

$$q = |(133757945)^2 - 766(4832871)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 13$, $p = 17$ and $q = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{766}\right]$ is a unique factorization domain.

If $d = 838$, then $t = 20$ and $\Gamma = \{3\}$. Furthermore, we get that

$$p = |(29)^2 - 838(1)^2|,$$

where $p = 3$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{838}\right]$ is a unique factorization domain.

If $d = 862$, then $t = 20$ and $\Gamma = \{3, 7, 11, 13, 19\}$. Furthermore, we get that

$$p = |(38667991)^2 - 862(1317037)^2|, \ \ p' = |(752219211)^2 - 862(25620688)^2|,$$

$$p'' = |(3613227)^2 - 862(123067)^2|, \ \ p''' = |(7017)^2 - 862(239)^2| \quad \text{and}$$

$$q = |(6056421679)^2 - 862(206282541)^2|,$$

where $p = 3$, $p' = 7$, $p'' = 11$, $p''' = 13$ and $q = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{862}\right]$ is a unique factorization domain.

If $d = 878$, then $t = 20$ and $\Gamma = \{11, 19\}$. Furthermore, we get that

$$p = |(563)^2 - 878(19)^2| \quad \text{and} \quad p' = |(89)^2 - 878(3)^2|,$$

where $p = 11$ and $p' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{878}\right]$ is a unique factorization domain.

If $d = 886$, then $t = 21$ and $\Gamma = \{3, 5, 7, 17\}$. Furthermore, we get that

$$p = |(26533757)^2 - 886(891419)^2|, \ \ p' = |(2105568673)^2 - 886(70737963)^2|,$$

$$p'' = |(325922986857)^2 - 886(10949596886)^2| \quad \text{and} \quad p''' = |(1514541)^2 - 886(50882)^2|,$$

where $p = 3$, $p' = 5$, $p'' = 7$ and $p''' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{886}\right]$ is a unique factorization domain.

If $d = 926$, then $t = 21$ and $\Gamma = \{5, 7, 13, 17\}$. Furthermore, we get that

$$p = |(213)^2 - 926(7)^2|, \ \ p' = |(2617)^2 - 926(86)^2|,$$

$$p'' = |(3754271)^2 - 926(123373)^2| \quad \text{and} \quad p''' = |(61)^2 - 926(2)^2|,$$

where $p = 5$, $p' = 7$, $p'' = 13$ and $p''' = 17$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{926}\right]$ is a unique factorization domain.

If $d = 958$, then $t = 21$ and $\Gamma = \{3, 11, 13\}$. Furthermore, we get that

$$p = |(31)^2 - 958(1)^2|, \ \ p' = |(1095531)^2 - 958(35395)^2| \quad \text{and}$$

$$p'' = |(12783)^2 - 958(413)^2|,$$

where $p = 3$, $p' = 11$ and $p'' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{958}\right]$ is a unique factorization domain.

If $d = 974$, then $t = 22$ and $\Gamma = \{5, 7, 13, 19\}$. Furthermore, we get that

$$p = |(2091)^2 - 974(67)^2|, \ \ p' = |(495973)^2 - 974(15892)^2|,$$

$$p'' = |(31)^2 - 974(1)^2| \quad \text{and} \quad p''' = |(156513)^2 - 974(5015)^2|,$$

where $p = 5$, $p' = 7$, $p'' = 13$ and $p''' = 19$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{974}\right]$ is a unique factorization domain.

If $d = 998$, then $t = 22$ and $\Gamma = \{7, 13\}$. Furthermore, we get that

$$p = |(695)^2 - 998(22)^2| \quad \text{and} \quad p' = |(6413)^2 - 998(203)^2|,$$

where $p = 7$ and $p' = 13$. By Proposition 3, we get that $\mathbb{Z}\left[\sqrt{998}\right]$ is a unique factorization domain.

$\square$

LEMMA 1. *Let $d \equiv 2, 3 \pmod 4$ be a positive integer and p, a, b, c and x_0 any integers, where p is prime. Suppose that $b^2 - 4ac = u^2 d$, for some integer $u \geq 1$. If $p \in \Delta(d)$ and $p \nmid a$, then there exists $n \in \mathbb{Z}$ such that*

$$x_0 \leq n \leq x_0 + p - 1, \qquad and \qquad an^2 + bn + c \equiv 0 \pmod p.$$

Proof. Since $p \in \Delta(d)$, we get that there exists $t \in \mathbb{Z}$ such that

$$(17) \qquad\qquad\qquad t^2 \equiv d \pmod p.$$

Now, we distinguish the two cases: $p \neq 2$ and $p = 2$. In the first case we get

$$(18) \qquad\qquad\qquad p \nmid 2a;$$

from (18) we deduce that there exists $w \in \mathbb{Z}$ such that

$$(19) \qquad 0 \leq w \leq p - 1 \qquad and \qquad 2aw \equiv ut - (b + 2ax_0) \pmod p.$$

Let $n = w + x_0$. Then, from (19) we obtain

$$(20) \qquad x_0 \leq n \leq x_0 + p - 1 \qquad and \qquad 2an + b \equiv ut \pmod p.$$

Since $b^2 - 4ac = u^2 d$, from (20) and (17), we deduce that

$$(21) \qquad \begin{aligned} 4a(an^2 + bn + c) &\equiv (2an + b)^2 - (b^2 - 4ac) \\ &\equiv (2an + b)^2 - u^2 d \equiv (ut)^2 - u^2 d \equiv 0 \pmod p. \end{aligned}$$

Combining (18) and (21), we get

$$an^2 + bn + c \equiv 0 \pmod p.$$

In the second case when $p = 2$, we get that a is odd. Since $d \equiv 2, 3 \pmod 4$ and $b^2 - 4ac = u^2 d$ it follows that bc is even. Therefore $ax_0^2 + bx_0 + c \equiv 0 \pmod 2$ or $a(x_0 + 1)^2 + b(x_0 + 1) + c \equiv 0 \pmod 2$. $\square$

The following consequence of Theorem 1 is interesting in its own right.

THEOREM 6. *Let $d \equiv 2, 3 \pmod 4$ be a non-square positive integer and a, b, c, x_0 any integers with $a > 0$. Also let $\Lambda(a)$ be the set of all primes $p \in \mathbb{N}$ satisfying $p \mid a$. Suppose that $b^2 - 4ac = u^2 d$, for some integer $u \geq 1$. Also suppose that the equation*

$$p = |x^2 - dy^2|$$

is solvable in integers x, y, for every $p \in \Lambda(a)$. If $|an^2 + bn + c|$ is 1 or prime for every integer n with $x_0 \leq n \leq x_0 + \max(\Delta(d)) - 1$, then $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.

Proof. Let $t \in \Delta(d)$. Let us see that there exist integers x and y such that

$$4t = |x^2 - dy^2|.$$

Since the equation $p = |x^2 - dy^2|$ is solvable in integers x, y, for every $p \in \Lambda(a)$ we assume, without loss of generality, that $t \nmid a$. Now, since $b^2 - 4ac = u^2 d$ and $t \in \Delta(d)$, by Lemma 1, there exists $n \in \mathbb{Z}$ such that

$$(22) \qquad x_0 \le n \le x_0 + t - 1$$

and

$$(23) \qquad an^2 + bn + c \equiv 0 \pmod{t}.$$

From (22), we get

$$x_0 \le n \le x_0 + \max(\Delta(d)) - 1,$$

and so, according to our hypotheses $|an^2 + bn + c|$ is 1 or prime. Thus, from (23) we get

$$(24) \qquad t = |an^2 + bn + c|.$$

From (24) we deduce that

$$4at = |(2an + b)^2 - (b^2 - 4ac)| = |(2an + b)^2 - du^2|$$

Since $d \equiv 2, 3 \pmod{4}$ we get that $\beta \in \mathbb{Z}[\alpha]$ and $at = |N(\beta)|$, where $\alpha = \sqrt{d}$ and $\beta = \frac{2an+b}{2} + \frac{u}{2}\sqrt{d}$. As for every $q \in \Lambda(a)$ the equation $q = |x^2 - dy^2| = |N(x + y\alpha)|$ is solvable in integers x, y, proceeding by induction we deduce, by [9, Theorem 3], that there exists $\gamma \in \mathbb{Z}[\alpha]$ such that $t = |N(\gamma)|$. Let us put $\gamma = r + s\alpha$. Then $t = |N(\gamma)| = |r^2 - ds^2|$. Consequently, the equation $t = |x^2 - dy^2|$ is solvable in integers x, y, for every $t \in \Delta(d)$. It follows then, by Theorem 1, that $\mathbb{Z}[\alpha]$ is a unique factorization domain which completes the proof. $\square$

We now apply Theorem 6 to provide a test for trivial class number of real quadratic fields.

COROLLARY 2. *Let d be a positive integer and b, c, x_0 any integers. Suppose that d is a prime $\equiv 3 \pmod{4}$ or $d = 2q$ with q a prime $\equiv 3 \pmod{4}$. Also suppose that $b^2 - 8c = u^2 d$, for some integer $u \ge 1$. If $|2n^2 + bn + c|$ is prime or equal to 1 whenever $x_0 \le n \le x_0 + \max(\Delta(d)) - 1$, then $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

Proof. By [14, Lemma 2.2 and Lemma 2.3] we get that the equation $2 = |x^2 - dy^2|$ is solvable in integers x, y. Therefore, corollary is obtained from Theorem 6. $\square$

Hence from Corollary 2 we obtain the following results:

PROPOSITION 4. *Let u and x_0 be integers, where u is odd. Suppose that $d = 2q$ where q is a prime congruent to 3 (mod 4), and that $|2n^2 - u^2 q|$ is prime or equal to 1 whenever $x_0 \le n \le x_0 + \max(\Delta(d)) - 1$. Then $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

PROPOSITION 5. *Let u, x_0 be integers, where u is odd. Suppose that d is a prime congruent to $3 \pmod 4$, and that $|2n^2 + 2n + \frac{1-u^2 d}{2}|$ is prime or equal to 1 whenever $x_0 \leq n \leq x_0 + \max(\Delta(d)) - 1$. Then $\mathbb{Z}[\sqrt{d}]$ is a unique factorization domain.*

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